Manifolds with non-negative Ricci curvature and Nash inequalities

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Abstract. We prove that for any complete n-dimensional Riemannian manifold with nonnegative Ricci curvature, if the Nash inequality is satisfied, then it is diffeomrphic to R^n

1. Introduction. Let M be any complete n-dimensional ($n \geq 2$) Riemannian manifold with nonnegative Ricci curvature, $C_0^\infty(M)$ be the space of smooth functions with compact support in M. Denote by dv and ∇ the Riemannian volume element and the gradient operator of M, respectively.

It is well known in [1] that Ledoux showed that: If one of the following Sobolev inequalities is satisfied,

$$||f||_p \leq C_0||\nabla f||_q, \forall f \in C_0^\infty(M),\ 1 \leq q < n,\ \frac{1}{p} = \frac{1}{q} - \frac{1}{n}. \tag{1}$$

where C_0 is the optimal constant in R^n, denote $||f||_p$ by the L^p norm of function f ;then M is isometric to R^n.

The basic idea of Ledoux's result is to find a function in $C_0^\infty(M)$, then one can substitute it to (1) and obtain that $Vol(B(x_0, r)) \geq V_0(r)$,here $Vol(B(x_0, r))$ denote the volume of the geodesic ball $B(x_0, r)$ of radius r with center x_0,and $V_0(r)$ the volume of the Euclidean ball of radius r in R^n. Since the Ricci curvature of M is nonnagtive ,from Bishop's comparison theorem[2],we know that $Vol(B(x_0, r)) \leq V_0(r)$, so M is isometric to R^n.

Later Xia combined Ledoux's method with Cheeger and Colding's result[3],which is that given an integer $n \geq 2$,there exists a constant $\delta(n) > 0$ such that any n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and $Vol(B(x_0, r)) \geq$

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$(1-\delta(n))V_0(r)$ for some $x_0 \in M$ and all $r > 0$ is diffeomorphic to R^n. He proved that [4]: If one of the following sobolev inequalities is satisfied,

$$||f||_p \leq C_1||\nabla f||_q, \forall f \in C_0^\infty(M),\ 1 \leq q < n,\ \frac{1}{p} = \frac{1}{q} - \frac{1}{n}. \tag{2}$$

where the positive constant $C_1 > C_0$,then M is diffiomorphic to R^n.

This is a beautiful result.As we known,the Sobolev inequality(2) belongs to a general family of inequalities of the type

$$||f||_p \leq C||f||_s^\theta\ ||\nabla f||_q^{1-\theta}, \forall f \in C_0^\infty(M),\ \frac{1}{t} = \frac{\theta}{s} + \frac{1-\theta}{p}. \tag{3}$$

(see [5]).Inequality (2) corresponds to $\theta = 0$. When $q = r = 2$,and $\theta = 2/(n+2)$,it corresponds to the Nash inequality

$$\Big(\int |\ f\ |^2\ dv\Big)^{1+\frac{2}{n}} \leq C\Big(\int |\ f\ |\ dv\Big)^{\frac{4}{n}} \int |\ \nabla f\ |^2\ dv,\ f \in C_0^\infty(M), \tag{4}$$

[see(6)].So we may naturally ask whether or not there has analogous result for the Nash inequality? In this note ,we confirmed this problem.

MAIN THEOREM Let M be a complete n-dimensional Riemannian manifold with non-negative Ricci curvature. If the Nash inequality (4) *is satisfied with the positive constant C, then M is diffeomorphic to* R^n.

2. Proof of Main Theorem. Before showing this theorem, we must introduce a lemma about Scheon and Yau's cut-off function (see[7]or[8]), because we will use it.

LEMMA 2.1 Suppose $(M,\ g_{ij}(x))$ is an n-dimensional complete Riemannian manifold with non-negative Ricci curvature. Then there exists a constant $\tilde{C}$ depending only on the dimension n, such that for any $x_0 \in M$ and any number $0 < r < +\infty$, there exists a smooth function $\varphi(x) \in C^\infty(M)$ satisfying

$$\begin{cases} e^{\tilde{C}(1+\frac{d(x,\ x_0)}{r})} & \leq \varphi(x) \leq e^{-(1+\frac{d(x,\ x_0)}{r})} \\ \quad |\ \nabla\varphi(x)\ | & \leq \frac{\tilde{C}}{r}\varphi(x) \end{cases} \tag{5}$$

for $\forall x \in M$, where $d(x,\ x_0)$ denote the distance between x and x_0 with respect to the metric $g_{ij}(x)$.

PROOF. In [7], Scheon and Yau constructed a C^∞ function $\psi(x)$ satisfying

$$\begin{cases} \frac{1}{\tilde{C}}(1+d(x,\ x_0)) & \leq \psi(x) \leq 1 + d(x,\ x_0) \\ \quad |\ \nabla\psi(x)\ | & \leq \tilde{C} \end{cases}$$

for $\forall x \in M$ and some positive constant $\tilde{C}$ depending only on the dimension n. Now we define a new metric on M by

$$\tilde{g}_{ij}(x) = \frac{1}{r^2}g_{ij}(x), \quad x \in M.$$

Then the new metric $\tilde{g}_{ij}(x)$ is still a complete Riemannian metric on M with non-negative Ricci curvature. Thus there exists a smooth function $h(x) \in C^{\infty}(M)$ such that

$$\begin{cases} \frac{1}{\tilde{C}}(1+\frac{d(x,x_0)}{r}) & \leq h(x) \leq -(1+\frac{d(x,x_0)}{r}) \\ \mid \tilde{\nabla} h(x) \mid_{\tilde{g}_{ij}(x)} & \leq \tilde{C} \end{cases}$$

for $\forall x \in M$. Here $\tilde{\nabla}$ and $| \cdot |_{\tilde{g}_{ij}(x)}$ are the gradient operator and norm with respect to the new metric $\tilde{g}_{ij}(x)$. Then by setting $\varphi(x) = e^{-h(x)}$ we get the desired cut-off function (4).

As following Ledoux's method,we want to look for a function in $C_0^{\infty}(M)$. we just obtain a smooth function from Lemma 2.1 ,if we substitute it to (4),then through direct compute we can get that $Vol(B(x_0, r)) \geq CV_0(r)$ $(0 < C < 1)$. From Cheeger and Colding's result ,we can prove the Main Theorem. Now the question is that $\varphi(x)$ hasn't compact support in M. To solve this problem, we can find a sequence functions $\varphi_m(x) \in C_0^{\infty}(M)$ such that $||\varphi_m||_t \to ||\varphi||_t$ and $||\nabla\varphi_m||_t \to ||\nabla\varphi||_t$, when $m \to +\infty$,for any $t > 0$. Then from Lebesgue dominated convergence theorem,we get $\varphi(x)$ is satisfied with the inequality (4).

PROPOSITION 2.1 For $\varphi(x)$ in (4), there exist a sequence functions $\varphi_m(x) \in C_0^{\infty}(M)$ such that $||\varphi_m||_t \to ||\varphi||_t$ and $||\nabla\varphi_m||_t \to ||\nabla\varphi||_t$, when $m \to +\infty$,for any $t > 0$.

PROOF. As we known, there always exists functions $\psi_m(x) \in C_0^{\infty}(M)$ such that $\psi_m(x) = 1$ for $x \in B(x_0, 2^m r)$,any fix positive number r, $\psi_m(x) = 0$ for $x \in M \setminus B(x_0, 2^{m+1}r)$, otherwise $0 \leq \psi_m(x) \leq 1$;and $|\nabla\psi_m| \leq \frac{2}{2^m r}$. Let $\varphi_m(x) = \psi_m(x)\varphi(x)$, then $\varphi_m(x) \in C_0^{\infty}(M)$.Thus

$$||\varphi_m - \varphi||_t^t = \int_M |\psi_m\varphi - \varphi|^t dv = \int_{M\setminus B(x_0,2^m r)} |\psi_m\varphi - \varphi|^t dv \leq \int_{M\setminus B(x_0,2^m r)} |\varphi|^t dv$$

then we only need to prove that

$$\int_{M\setminus B(x_0,2^m r)} |\varphi|^t dv \longrightarrow 0, when\ m \longrightarrow +\infty. \tag{5}$$

From Lemma 2.1,we have

$$\int_{M\setminus B(x_0,2^m r)} |\varphi|^t dv \leq \int_{M\setminus B(x_0,2^m r)} exp(-t(1+\frac{d(x_0,x)}{r})dv$$

$$\leq C\int_{2^m r}^{+\infty} e^{-\frac{t}{r}\rho}\rho^{n-1}d\rho$$

$$\leq -C\rho^{n-1}e^{-\frac{t}{r}\rho} \mid_{2^m r}^{+\infty} + C\int_{2^m r}^{+\infty} e^{-\frac{t}{r}\rho}\rho^{n-2}d\rho, (m \longrightarrow +\infty)$$

$$\leq C\int_{2^m r}^{+\infty} e^{-\frac{t}{r}\rho}\rho^{n-2}d\rho$$

$$\leq ... \leq C\int_{2^m r}^{+\infty} e^{-\frac{t}{r}\rho}d\rho$$

$$\leq -Ce^{-\frac{t}{r}\rho} \mid_{2^m r}^{+\infty} \longrightarrow 0, (m \longrightarrow +\infty).$$

So we obtain that $||\varphi_m||_t \to ||\varphi||_t, m \longrightarrow +\infty$. From (4) and $|\nabla\psi_m| \leq \frac{2}{2^m r}$, we get that

$$||\nabla\varphi_m - \nabla\varphi||_t = ||\nabla\psi_m\varphi + \psi_m\nabla\varphi - \nabla\varphi||_t$$

$$\leq ||\nabla\psi_m\varphi||_t + ||(\psi_m - 1)\nabla\varphi||_t$$

$$\leq (\int_{M\backslash B(x_0,2^m r)} |\nabla\psi_m|^t|\varphi|^t dv)^{\frac{1}{t}} + (\int_{M\backslash B(x_0,2^m r)} |\psi_m - 1|^t|\nabla\varphi|^t dv)^{\frac{1}{t}}$$

$$\leq \frac{2}{2^m r}(\int_{M\backslash B(x_0,2^m r)} |\varphi|^t dv)^{\frac{1}{t}} + \frac{\tilde{C}}{r}(\int_{M\backslash B(x_0,2^m r)} |\varphi|^t dv)^{\frac{1}{t}}$$

Which combining with (5) implies that $||\nabla\varphi_m||_t \to ||\nabla\varphi||_t$, $m \to +\infty$.

Now we prove the Main Theorem.

PROOF: By Proposition 2.1 we know the function $\varphi(x)$ satisfies with the Nash inequality.Together with (4) we get

$$\left(\int | \varphi |^2 dv\right)^{1+\frac{2}{n}} \leq C\left(\int | \varphi | dv\right)^{\frac{4}{n}}\left(\int | \nabla\varphi |^2 dv\right)$$

$$\left(\int | \varphi |^2 dv\right)^{\frac{2}{n}} \leq C\left(\int | \varphi | dv\right)^{\frac{4}{n}}\left(\frac{\tilde{C}^2}{r^2}\int | \varphi |^2 dv\right)$$

$$\left(\int | \varphi |^2 dv\right)^{\frac{2}{n}} \leq \frac{C\tilde{C}^2}{r^2}\left(\int | \varphi | dv\right)^{\frac{4}{n}}$$

$$\left(\int_M | \varphi |^2 dv\right) \leq \left(\frac{C\tilde{C}^2}{r^2}\right)^{\frac{n}{2}}\left(\int_M | \varphi | dv\right)^2$$

$$\int_M e^{-2\tilde{C}\left(1+\frac{d(x,\,x_0)}{r}\right)} dv \leq \left(\frac{C\tilde{C}^2}{r^2}\right)^{\frac{n}{2}}\left(\int_M e^{-\left(1+\frac{d(x,\,x_0)}{r}\right)} dv\right)^2$$

$$\int_{B(x_0,\,r)} e^{-2\tilde{C}\left(1+\frac{d(x,\,x_0)}{r}\right)} \leq \left(\frac{C\tilde{C}^2}{r^2}\right)^{\frac{n}{2}}\left[vol(B(x_0,\,r)) + \sum_{k=0}^{+\infty}\int_{B(x_0,\,2^{k+1}r)\backslash B(x_0,\,2^k r)} e^{-2\left(1+\frac{d(x,\,x_0)}{r}\right)}\right]^2$$

$$e^{-4\tilde{C}}vol(B(x_0,\,r)) \leq \left(\frac{C\tilde{C}^2}{r^2}\right)^{\frac{n}{2}}\left[vol(B(x_0,\,r)) + \sum_{k=0}^{+\infty} e^{-2^k}(2^{k+1})^{2n} vol(B(x_0,\,r))\right]^2$$

$$e^{-4\tilde{C}}vol(B(x_0,\,r)) \leq \left(\frac{C\tilde{C}^2}{r^2}\right)^{\frac{n}{2}} C_2^2(vol(B(x_0,\,r)))^2$$

$$vol(B(x_0,\,r)) \geq e^{-4\tilde{C}}(C\tilde{C}^2)^{-\frac{n}{2}}C_2^{-2}r^n$$

Let $C = e^{-4\tilde{C}}(C\tilde{C}^2)^{-\frac{n}{2}}C_2^{-2}$, where $C_2 = 1 + \sum_{k=0}^{+\infty} e^{-2^k}(2^{k+1})^{2n}$, then $vol(B(x_0,\,r)) \geq Cr^n$.

Then from above discussion,we know that there exists a number $\delta(n)$,($0 < \delta(n) < 1$),such that $vol(B(x_0,\,r)) \geq (1-\delta(n))V_0(r)$,together with Cheeger and Colding's result, which implies M is diffeomorphic to R^n. The proof of the Main Theorem is completed.

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